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6 A PARAMETRIC ALGORITHM FOR DRAWING PICTURES OF SOLID OBJECTS BOUNDED BY QUADRIC SURFACES.

by

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ABSTRACT

An algorithm is described for generating two-dimensional, visible-line projections of three-dimensional objects that are bounded by patches of quadric surfaces.

The main task of the algorithm is the calculation of intersections between quadric surfaces. A parameterization scheme is used. Each quadric-surface intersection curve (QSIC) is represented as a set of coefficients and parameter limits. Each value of the parameter represents at most two points, and these may easily be distinguished. This scheme can find the coordinates of points of even quartic (fourth-order) intersection curves, using equations of no more than second order.

Methods of parameterization for each type of QSIC are discussed, as well as the problems of surface bounding and hidden-surface removal.

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I. INTRODUCTION

1.1 General

This report describes an algorithm for the computer generation of orthographic as well as perspective projections of three-dimensional objects whose surfaces are made up of segments of quadric surfaces, also known as quadric patches.

There are now many algorithms for generating pictures of solid objects whose surfaces consist of planar polygons. A polygon is a segment of a first-order surface (plane) bounded by other first-order surfaces.

The Braid algorithm [Braid 1975] extends this to allow also cylinders and parts of cylinders. Cylinders are quadric surfaces.

This paper is based on the Woon algorithm [Woon 1970 and Woon-Freeman 1971], which processes second-order (quadric) surfaces bounded by second-order surfaces. Quadric surfaces include such objects as spheres, cylinders, cones, ellipsoids and hyperbolic paraboloids. They are not too complex mathematically.

An important advantage of using quadric patches is that many objects (especially man-made ones) can be more precisely modelled by a small number of quadric patches than by a large set of small polygons.

Another advantage is that shaded pictures can be made with better control over such effects as mach banding and specular reflection [Phong 1975].

Previous quadric patch algorithms, such as [Mahl 1972] and [Weiss 1966], as well as the Woon algorithm, become laborious when solving for the intersection of two quadric surfaces. This is a fourth-order problem, and there is no easy method for finding the roots of a fourth-order equation, or even merely determining whether there are any real roots.

The algorithm presented here reduces each fourth-order problem into a second-order problem. The algorithm uses a parametric representation. Every point on the intersection of two quadric surfaces is determined uniquely by a numerical parameter and, in many (but not all) cases, by a Boolean sign parameter. Since the fourth-order quadric surface intersection curves (QSICs) can be represented parametrically, the computer memory need only store about a score of numbers to represent a QSIC, rather than storing some lengthy point-by-point description for each QSIC.

1.2 The Woon Algorithm

Since the Woon algorithm is not well known, and since it forms a basis of this algorithm, the Woon algorithm will be presented first.

For each object, the input data for the Woon algorithm consists of the surface equations, surface bounds, surface intersections, and vertices. Once an object is specified, the user may ask for different views of the object, representing different locations of the vantage point. Pictures may be generated in either orthographic or perspective projection, and with the hidden lines displayed, dashed, or suppressed.

For each surface the user must specify the following:

a) The equation of the surface. This is expressed in terms of the ten coefficients of the equation:

$$q(x,y,z) = q_1x^2 + q_2y^2 + q_3z^2 + q_4xy + q_5yz + q_6zx + q_7x + q_8y + q_9z + q_0 = 0 \qquad (1)$$

The quadric surface divides the three-dimensional space into two open regions: the exterior, in which $q(x,y,z)>0$; and the interior, in which $q(x,y,z)<0$. On the surface itself, $q(x,y,z)=0$.

b) The bounds of the surface. When using the Woon algorithm, one is normally interested in only one connected segment of each surface. This surface segment (the "quadric patch") is defined as the locus of points on the surface satisfying a certain Boolean condition, which is specified for each surface.

Each bound consists of another quadric surface, and a polarity (+ or -). If the polarity is positive (+), the bound is satisfied (true) only for points in the exterior of the bounding surface. If the polarity is negative (-), the bound is satisfied only for points in the interior of the bounding surface.

Each set of bounds consists of one or more bounds. The set of bounds is satisfied if and only if all bounds in the set are satisfied. Each patch has one or more sets of bounds. The patch is the locus of all points on the surface satisfying at least one set of bounds.

c) The intersection of the surface patch with other surface patches. Woon distinguishes between planar surface interesections which lie in a plane, and non-planar intersections which do not. He also distinguishes between protrusive and recessive intersections.

In addition to the above information, needed for each surface and each surface intersection, the user must also specify all real vertices. These are points where three or more intersection curves meet.

Quadric surfaces have the property, when viewed from certain viewpoints, of "folding in back of themselves", such that part of the quadric surface hides another part. The curve at which this happens, the locus of points of tangency of rays from the vantage point to the surface, is called a "limb". (Astronomers use this term for the visual edge of a celestial body.)

The drawing produced by the Woon algorithm consists of edges. There are two types: <u>real</u> edges, which are portions of the intersections of quadric surfaces; <u>virtual</u> edges, which are portions of limbs.

Woon used a variation of the Loutrel algorithm [Loutrel 1970] to determine whether or not points are visible. This requires that the quadric patches be broken up into sections called <u>faces</u>. A patch which folds in back of itself is divided into two or more faces by the limb.

A face is classified as a <u>front face</u> or a <u>back face</u>, depending upon its orientation with respect to a viewer. Only front faces can be visible. The orientation of the faces forming an edge, as well as the protrusiveness or recessiveness of the edge, help determine whether the edge is potentially visible.

The computation of pictures is divided into two parts. First, for each object, the object-dependent quantities are computed. These include real edges, real vertices, and the partial classification of edges.

Then, for each view of the object, the virtual edges and virtual vertices (intersections of virtual edges and real edges) are computed, and finally the visible portions of each edge are found. To generate the data for drawing the edges, the object must be rotated about three angles: azimuth (ϕ), elevation (θ), and twist (Ψ). For perspective drawings, the distance from the origin to the viewpoint (D) and the distance from the viewpoint to the picture plane (d) must be considered.

Woon suggests two methods for calculating points on a quadric surface intersection curve (QSIC). If the QSIC is planar it will be a conic section, and will be relatively easy to handle mathematically. If the QSIC is not planar, then the curve may be computed as a series of points, each within a specified distance from its neighbors. The term "vector" is used for the directed line segment from one of these points to the next.

Each vector has a dominant direction, either $\pm x$, $\pm y$, or $\pm z$. For the first vector in a curve, a guess is made as to its dominant direction, and, if it takes the curve out-of-bounds for its surfaces, then it is rejected and another direction is selected. For subsequent vectors, a strong hint as to the dominant direction is provided by the direction of the last vector. Newton's method is then used to obtain the best approximation to the actual direction.

Woon discusses a method of determining whether an intersection is planar or not. This method is discussed and expanded upon later in this paper.

The Loutrel algorithm [Loutrel 1970] is used by Woon to determine whether or not an edge or point is visible. Each point on an edge has an order of invisibility, which is the number of front-faces between it and the vantage point. The order of invisibility changes whenever the line from the point to the vantage point intersects an edge of the hiding surface.

Loutrel's original algorithm dealt only with polygonal faces. All edges were straight-line segments; computing the intersecting points of edge projections was exceedingly simple. This is not the case for the projections of edges of quadric patches. These edges may be as high as fourth-order curves. Woon specifies a very complicated algorithm for determining these intersection points, as well as an algorithm for propagating the order of invisibility from edge to edge. None of this will be used in this report.

1.3 QUADRAW

Woon wrote a program called QUADRAW which implements a variation of his algorithm. For each object, QUADRAW requires information on scaling, vector length, and tolerances.

For each surface, QUADRAW requires the following:

(1) The equation of the surface. (There are also some "auxiliary surfaces" which are not part of the object, but bound other surfaces.)

(2) The bounds of the surface.

For each surface intersection, it requires:

(1) The two intersecting surfaces.

(2) The type (recessive or obtrusive).

(3) The planarity of the intersecting surfaces and of the surface intersection.

(4) If neither surface is planar, but the intersection is, the equation of the plane.

(5) Specification of each real vertex.

QUADRAW computes all curves in a piece-wise linear fashion. Each QSIC is represented in memory by a series of small vectors. This approach requires use of a computer with a large primary memory.

QUADRAW does not use Woon's variation of the Loutrel algorithm to its fullest extent. It does use classification of edges to determine which edges might be visible, but then uses a brute-force method to compute edge visibility. This brute-force method works nicely, and is used here, with some modification.

As each view is processed, the maxima and minima in the view-plane are computed for each surface. For each potentially visible edge, visibility is computed one point at a time. Each point is checked against each face whose extrema include the coordinates of the point. If at least one face is found which hides the point, then the point is invisible.

The testing to see whether a face hides a point on the edge (test point) proceeds as follows:

The line of sight is computed from the vantage point to the test point. The equation of the surface is reduced by considering only points along the line of sight. This results in a second-order equation is most cases, with first-order equations sometimes occurring. If the solution of the equation yields two roots, one corresponds to a point on a back face and may be discarded. If the remaining point is within bounds, then it hides the test point, and no further test is needed for that point.

If there is no surface hiding the test point, then it is visible.

The method suggested later on in this report is a refinement of this method.

II. THE QUADRATIC FORM

The quadratic surface, as represented by eq. (1) is a quadratic form [Woon 1970, Dresden 1930, Newman-Sproull 1973]. It may be represented in vector-matrix form, as follows:

Suppose the location of a point in 3-space is represented by the vector: $\vec{x} = (x\ y\ z\ 1)$. Eq. (1) then may be represented as follows:

$$q(x,y,z) = q(\vec{x}) = (x\ y\ z\ 1)\begin{bmatrix} q_1 & \tfrac{1}{2}q_4 & \tfrac{1}{2}q_6 & \tfrac{1}{2}q_7 \\ \tfrac{1}{2}q_4 & q_2 & \tfrac{1}{2}q_5 & \tfrac{1}{2}q_8 \\ \tfrac{1}{2}q_6 & \tfrac{1}{2}q_5 & q_3 & \tfrac{1}{2}q_9 \\ \tfrac{1}{2}q_7 & \tfrac{1}{2}q_8 & \tfrac{1}{2}q_9 & q_0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{2}$$

or, for short,

$$q(\vec{x}) = \vec{x}\ Q\ x^T \tag{3}$$

where Q is the 4 x 4 matrix in equation (2). Q is called the discriminant of the quadric surface. Notice that Q is symmetric and that, for any real non-zero scalar β, βQ is equivalent to Q, since they describe the same surface. In this report the same symbol will on occasion be used for both a quadric surface as well as its discriminant.

An alternate form which will also be used is the following:

$$Q = \begin{bmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{bmatrix} \tag{4}$$

such that

$$q(x,y,z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2\,Fzx$$
$$+ 2Gx + 2Hy + 2Jz + K = 0 \tag{5}$$

The upper-left 3 x 3 principal sub-matrix of the discriminant contains all the second-order terms. This will be called the "sub-discriminant".

$$Q_u = \begin{bmatrix} q_1 & \tfrac{1}{2}q_4 & \tfrac{1}{2}q_6 \\ \tfrac{1}{2}q_4 & q_2 & \tfrac{1}{2}q_5 \\ \tfrac{1}{2}q_6 & \tfrac{1}{2}q_5 & q_3 \end{bmatrix} = \begin{bmatrix} A & D & F \\ D & B & E \\ F & E & C \end{bmatrix} \tag{6}$$

The sub-discriminant will always be represented by the subscript u.

The rank of the discriminant and the sub-discriminant are helpful in classifying quadric surfaces. These are invariant under the transformation described below. The rank of the discriminant is invariant under any nonsingular transformation, and that of the sub-discriminant is invariant under any transformation in which the upper-left 3 x 3 sub-matrix is nonsingular.

2.1 Transformations of the Discriminant

In setting up the description of an object, the <u>untransformed</u> space will be the u-v-w space, and the <u>transformed</u> space will be the x-y-z space. The transformation matrices will be F and $\bar{F}$, each being the inverse of the other.

First let us consider the tranformation $\vec{x} = \vec{u}F$, where $\vec{x} = (x\ y\ z\ 1)$ and $\vec{u} = (u\ v\ w\ 1)$, let

$$F = \begin{bmatrix} f_{ux} & f_{uy} & f_{uz} & 0 \\ f_{vx} & f_{vy} & f_{vz} & 0 \\ f_{wx} & f_{wy} & f_{wz} & 0 \\ \delta_x & \delta_y & \delta_z & 1 \end{bmatrix} \tag{7}$$

where F is a congruence transformation. The sub-discriminant, F_u, is an orthogonal transformation. If F_u is orthonormal, then the determinants of both F_u and F are unity.

Assume that $\mathcal{F} = F^{-1}$. $\mathcal{F}_u$ is the transpose of F_u if they are orthonormal. The lower-right element of $\mathcal{F}$ (the constant term) is unity.

Suppose the equation for the quadric surface P is:

$$\vec{u}\, P\, \vec{u}^T = 0 \quad \text{in untransformed space, and} \tag{8a}$$

$$\vec{x}\, \mathcal{P}\, \vec{x}^T = 0 \quad \text{in transformed space.} \tag{8b}$$

Using the transformation $\vec{x} = \vec{u}F$ on equation (8b), we have: $\vec{u}\, F\, \mathcal{P}\, F^T\, \vec{u}^T$. Comparing this with equation (8a), we have

$$P = F\, \mathcal{P}\, F^T \tag{9}$$

By pre- and post-multiplication, we have:

$$\mathcal{F}\, P\, \mathcal{F}^T = \mathcal{F}\, F\, \mathcal{P}\, F^T\, \mathcal{F}^T = I\, \mathcal{P}\, I = \mathcal{P}, \quad \text{or} \quad \mathcal{P} = \mathcal{F}\, P\, \mathcal{F}^T \tag{10}$$

It is possible to break down the transformation into two parts: <u>rotation</u> and <u>translation</u>. They may be applied repeatedly in any order.

The rotation matrix is:

$$R = \begin{bmatrix} f_{ux} & f_{uy} & f_{uz} & 0 \\ f_{vx} & f_{vy} & f_{vz} & 0 \\ f_{wx} & f_{wy} & f_{wz} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

The sub-dsicriminant is affected only by rotation. Rotation does not affect the constant term, in the lower-right corner of the discriminant.

The translation matrix is:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta_x & \delta_y & \delta_z & 1 \end{bmatrix} \qquad (12)$$

Translation does not affect the sub-discriminant. The constant term is affected only by translation.

A transformation can be formed by a rotation followed by a translation ($F = RT$), or by a translation followed by a rotation ($F = TR$). T and R do not usually commute, just as rotation and translation to not usually commute.

2.2 Canonical Form

It is possible to put the discriminant of a surface into a canonical form, in which the axes of the coordinate system are the axes of the object, with non-zero diagonal elements of the discriminant being given precedence. In finding the canonical form, one also finds the transformation matrix which will transform the discriminant into its canonical form.

In the following algorithm, the discriminant P is transformed into its canonical form Q by transformation matrix $\mathcal{F}$. The matrix Σ is used as an intermediate transformation. The operation "$\leftarrow$" means assignment, as in a computer program.

a) First, a rotation is performed to diagonalize P_u. This rotation is represented as the orthonormal matrix $\mathcal{F}_u$. We have $Q_u \leftarrow \mathcal{F}_u P_u \mathcal{F}_u^T$.

If P_u has both non-zero and zero eigenvalues, the non-zero ones should be placed towards the upper-left corner of Q_u, and the zero ones towards the lower-right.

$\mathcal{F}$ is then formed by augmenting $\mathcal{F}_u$:

$$\mathcal{F} \leftarrow \left[\begin{array}{ccc|c} & & & 0 \\ & \mathcal{F}_u & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{13}$$

We have $Q \leftarrow \mathcal{F}\, P\, \mathcal{F}^T$, and Q is of the form

$$\begin{bmatrix} A & 0 & 0 & G \\ 0 & B & 0 & H \\ 0 & 0 & C & J \\ G & H & J & K \end{bmatrix}$$

b) We must now try to eliminate as many off-diagonal elements as possible. (Skip this step if $A = B = C = 0$.) We take Σ as:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \delta_x & \delta_y & \delta_z & 1 \end{bmatrix} \tag{14}$$

where

$\delta_x = -G/A;$

$\delta_y = -H/B$ if $B \neq 0$, $\qquad \delta_y = 0$ if $B = 0$;

$\delta_z = -J/C$ if $C \neq 0$, $\qquad \delta_z = 0$ if $C = 0$.

We next have the "standard" transformation:

$$P \leftarrow \Sigma\, P\, \Sigma^T, \qquad \mathcal{F} \leftarrow \Sigma\, \mathcal{F}. \tag{15a,b}$$

c) At this point P is in one of four forms

If P is in the form:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & K \end{bmatrix}$$

then it is already in canonical form.

If it is in the form:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & J & K \end{bmatrix}$$

then it is a paraboloid. See (d), below, on how to eliminate the K (constant) term.

If it is in one of these forms:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & 0 & H \\ 0 & 0 & 0 & J \\ 0 & H & J & K \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & G \\ 0 & 0 & 0 & H \\ 0 & 0 & 0 & J \\ G & H & J & K \end{bmatrix}$$

then we must remove all off-diagonal pairs except for the top- (left-) most one. This may be done as follows:

For the case on the left, one uses the transformation matrix:

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & K & S & 0 \\ 0 & -S & K & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (16)$$

with $K^2 + S^2 = 1$ and $\frac{C}{S} = \frac{H}{J}$

This removes one of the off-diagonal pairs, making J=0. One then uses the standard transformation (15a,b).

For the case on the right, this must be done twice, once to remove H, and once to remove J.

d) At this point, P has one off-diagonal element. If K = 0, we have the canonical form. If not, we make a translation of the same form as eq. (14) except with the following δ's:

$$\begin{aligned} \delta_x &= -K/2G \text{ if } G \neq 0, & \delta_x &= 0 \text{ if } G = 0; \\ \delta_y &= -K/2H \text{ if } H \neq 0, & \delta_y &= 0 \text{ if } H = 0; \\ \delta_z &= -K/J \text{ if } J \neq 0, & \delta_z &= 0 \text{ if } J = 0. \end{aligned} \tag{17}$$

After applying the standard transformation (15a,b), we have the final canonical form for P, and the final transformation matrix $\mathcal{F}$.

2.3 The Discriminant Form for Conic Sections

The conic sections are quadric curves, being also of the quadratic form. The discriminants of a conic are 3 x 3 matrices, and the sub-discriminants are the 2 x 2 upper-left submatrices of the discriminants. Other than the reduction in the number of dimensions, the conic sections are completely analogous to quadric surfaces; the transformation matrices are slightly simpler:

$$F = \begin{bmatrix} f_{ux} & f_{uy} & 0 \\ f_{vx} & f_{vy} & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix} = \begin{bmatrix} K & S & 0 \\ -S & K & 0 \\ \delta_x & \delta_y & 1 \end{bmatrix} \tag{18}$$

where $K^2 + S^2 = 1$

2.4 Classification of Quadric Surfaces and Conic Sections

Tables I and II are guides to the classification of quadric surfaces and conic sections, respectively. They are adapted, in part, from a chart given by Dresden [Dresden 1930, p. 230].

One of the columns in these tables is the absolute value of a signature. The signature of a matrix is the number of positive eigenvalues minus the number of negative eigenvalues.

In Table I, we use the following:

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{21} & q_{22} & q_{23} & q_{24} \\ q_{31} & q_{32} & q_{33} & q_{34} \\ q_{41} & q_{42} & q_{43} & q_{44} \end{bmatrix} = \begin{bmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{bmatrix} ; \qquad (19)$$

$$Q_u = \begin{bmatrix} A & D & F \\ D & B & E \\ F & E & C \end{bmatrix} \qquad (20)$$

$d = \mathrm{Rank}(Q)$; $\quad$ $m = \mathrm{Rank}(Q_u)$; $\quad$ $s = \mathrm{abs}(\mathrm{Signature}(Q_u))$.

$$T_1 = \sum_{i=1}^{3} q_{ii} = A + B + C \qquad (21)$$

$$T_2 = \sum_{i=1}^{3} \sum_{j=i+1}^{3} \begin{vmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{vmatrix} = AB + AC + BC - D^2 - E^2 - F^2 \qquad (22)$$

$$D_2 = \sum_{i=1}^{3} \sum_{j=i+1}^{4} \begin{vmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{vmatrix} = T_2 + T_1 K - G^2 - H^2 - J^2 \qquad (23)$$

$$D_3 = \sum_{i=1}^{2} \sum_{j=i+1}^{3} \sum_{k=j+1}^{4} \begin{vmatrix} q_{ii} & q_{ij} & q_{ik} \\ q_{ji} & q_{jj} & q_{jk} \\ q_{ki} & q_{kj} & q_{kk} \end{vmatrix} \tag{24}$$

$$\begin{aligned} = \; & ABC + ACK + ABK + BCK - 2\,(DEF + PGJ + DGH + EHJ) \\ & - D^2(C+K) - E^2(A+K) - F^2(B+K) - G^2(B+C) - H^2(A+C) - J^2(A+B) \end{aligned} \tag{25}$$

SINGULAR SURFACES

d	m	s	Conditions	Real surface (Imaginary part in parens.)
Planar				
1	0	0		INVALID
1	1	1		Coincident Planes
2	0	0		Single Plane
2	1	1	$D_2>0$	INVALID (Imag. Parallel Planes)
2	1	1	$D_2<0$	Two Parallel Planes
2	2	0	$T_2<0$	Two Intersecting Planes
2	2	2	$T_2>0$	LINE (Two Intersecting Imaginary Planes)
Non-planar Singular				
3	1	1		Parabolic Cylinder
3	2	0	$T_2<0$	Hyperbolic Cylinder
3	2	2	$T_2>0$; $T_1D_3<0$	Elliptic Cylinder
3	2	2	$T_2>0$; $T_1D_3>0$	INVALID (Imag. Cylinder)
3	3	1	α	Cone
3	3	3	β	POINT (Imaginary Cone)

NON-SINGULAR SURFACES $d=4$; $m=2$ if $\det(Q_u) = 0$

0 denotes zero; $m=3$ if $\det(Q_u) \neq 0$

+ denotes positive; - denotes negative; ± denotes nonzero.

$\lvert Q_u \rvert$	$\lvert Q \rvert$	s	Conditions	Surface
0	+	0	$(T_2<0)$	Hyperbolic Paraboloid
0	-	2	$(T_2>0)$	Elliptic Paraboloid
±	+	1	α	Hyperboloid of One Sheet
±	-	1	α	Hyperboloid of Two Sheets
±	+	3	β	INVALID (Imaginary Ellipsoid)
±	-	3	β	Ellipsoid

Conditions: α: $T_2>0$; $\det(Q_u) \times T_1 \leq 0$; or: $T_2 \leq 0$.

β: $T_2>0$; $\det(Q_u) \times T_1>0$.

Table I. Guide for the Classification of Quadric Surplus (adapted from Dresden 1930)

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} A & D & G \\ D & B & H \\ G & H & K \end{bmatrix} \quad ; \quad Q_u = \begin{bmatrix} A & D \\ D & B \end{bmatrix}$$

$d = \mathrm{Rank}(Q)$; $m = \mathrm{Rank}(Q_u)$; $s = \mathrm{abs}(\mathrm{Signature}(Q_u))$

$T_1 = q_{11} + q_{22} = A + B$

$$D_2 = \sum_{i=1}^{2} \sum_{j=i+1}^{3} \begin{vmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{vmatrix} = AB + BK + AK - D^2 - G^2 - H^2$$

d	m	s	Conditions	Curve
Singular				
1	0	0		INVALID
1	1	1		Coincident Lines
2	0	0		Single Line
2	1	1	$D_2 > 0$	INVALID (Imag. Parallel Lines)
2	1	1	$D_2 < 0$	Two Parallel Lines
2	2	0	$\lvert Q_u \rvert < 0$	Two Intersecting Lines
2	2	2	$\lvert Q_u \rvert > 0$	POINT (Inter. Imag. Lines)
Non-singular				
3	1	1		Parabola
3	2	0	$\lvert Q_u \rvert < 0$	Hyperbola
3	2	2	$\lvert Q_u \rvert > 0$; $\lvert Q \rvert T_1 < 0$	Ellipse
3	2	2	$\lvert Q_u \rvert > 0$; $\lvert Q \rvert T_1 > 0$	INVALID (Imaginary Ellipse)

Table II. Guide for the Classification of Conic Sections (adapted from Dresden 1930)

III. CLASSIFICATION OF QUADRIC SURFACE INTERSECTIONS

Given two quadric surfaces, the first two questions that arise are:

1) Do they intersect? and

2) If so, what is the nature of their quadric surface intersection curve (QSIC)?

3.1 The Pencil of Two Quadric Surfaces

Suppose we have two quadric surfaces, with discriminants P and Q. In matrix form, the equations for the two surfaces are $\vec{x}\, P\, \vec{x}^T = 0$ and $\vec{x}\, Q\, \vec{x}^T = 0$.

The equation

$$\vec{x}\,(Q-\alpha P)\vec{x}^T = 0 \tag{26}$$

represents, for all real values of α(finite or infinite), a surface on the "pencil" of P and Q.

For $\alpha = 0$, we have the surface Q. For $\alpha = \pm\infty$, we have the surface P.*

If surfaces P and Q intersect, then their intersection (QSIC) is the "base curve" of the pencil, and it lies in all the surfaces of the pencil. This "base curve" is not to be confused with the "base curve" of a parameterization surface, which will be used extensively later on.

If the two surfaces do not intersect, then none of the real surfaces of the pencil intersect. In addition, the pencil contains some imaginary surfaces, and these may be among those listed as INVALID in Table I.

---

* Since βP represents the same surface as P for any real non-zero scalar β; if $\beta=1/\alpha$, $\lim_{\alpha\to\infty} (1/\alpha)(Q-\alpha P) = \lim_{\alpha\to\infty} (Q/\alpha) - P = -P$, which is equivalent to P.

The general form for a member of the pencil of P and Q is given by

$$R(\alpha) = Q - \alpha P. \tag{27}$$

3.2 Classification of Pencils

One may classify pencils according to the classification of the simplest surface in the pencil. If the pencil has a base curve, then the base curve (a QSIC) has the same classification as its pencil.

If, for some α, $R(\alpha)$ has rank one or two, then $R(\alpha)$ represents either a plane or pair of planes. This pencil is called "planar".

Failing this, if, for some α, $R(\alpha)$ has rank three, the pencil is "non-planar singular".

If $R(\alpha)$ is never singular, then it always has rank four, and the pencil is non-singular.

Two surfaces do not intersect if one of the following occurs:

1) For some value of α, $R(\alpha)$ is INVALID; or

2) Either P or Q does not intersect some $R(\alpha)$.

3.3 Determining Whether a Surface Intersection is Planar or Singular

If, for some value of α, $R(\alpha)$ has rank of two or less, then the pencil is planar. As Woon has pointed out [Woon 1970, pp. 34-36], this occurs when two conditions are met for the same value of α:

(1) $\mathrm{Det}(R(\alpha)) = 0$; and

(2) The sum of the 3 x 3 principal minors of $R(\alpha)$ vanishes. This may be expressed as: $D_3(R(\alpha)) = 0$.

If condition (1) is met but condition (2) is not, then the surface is non-planar singular.

3.4 Non-singular Pencils

If $R(\alpha)$ is never singular, then the pencil is non-singular. The Appendix demonstrates that a non-singular QSIC must lie in a hyperbolic paraboloid.

IV. THEORY OF PARAMETERIZATION

In this section, methods of parameterization are discussed. The parameterization is done in a u-v-w coordinate system. A congruence transformation is then used to transform point coordinates into the x-y-z system. The parameter is denoted by t.

4.1 Parabola

Suppose we have a parabola of the form $Au^2 + 2Hv = 0$. Taking $m = -A/2H$, we have $v = mu^2$. The parameter equations are

$$u = t; \qquad v = mt^2. \tag{28}$$

4.2 Ellipse

Suppose we have an ellipse of form $Au^2 + Bv^2 + K = 0$. Taking $r_u = \sqrt{-K/A}$ and $r_v = \sqrt{-K/B}$ as being the semi-axes, we have

$$\frac{u^2}{r_u^2} + \frac{v^2}{r_v^2} - 1 = 0$$

The parametric equations are

$$u = \frac{2t}{1+t^2} r_u \qquad \text{and} \qquad v = \frac{1-t^2}{1+t^2} r_v. \tag{29}$$

This form is to be used only $-1 \leq t \leq +1$, in which case v will have only positive values. (A more complete method is given in section 4.4.)

This parameterization is well-behaved for values of t within the range [-1, +1]. That is, if $ds = \sqrt{du^2 + dv^2}$, ds/dt does not vary too much if r_u and r_v are of the same order of magnitude.

4.3 Hyperbola

Suppose we have a hyperbola of the form $Au^2 + Bv^2 + K = 0$. Taking $r_u = \sqrt{K/A}$ and $r_v = \sqrt{-K/B}$, we have

$$u = \frac{2t}{1-t^2} r_u; \qquad v = \frac{1+t^2}{1-t^2} r_v, \tag{30}$$

for $-1<t<+1$.

A hyperbola has two disjoint parts. This parameterization gives only one of them the one for which $v>0$.

In contrast to the parameterization of the ellipse, this form is not well-behaved. At values of t approaching ± 1, small changes in t result in large changes in u and v. However, the hyperbola is then very close to its asymptotes and is practically a straight line.

4.4 More Complete Forms

For central conics (ellipses and hyperbolae), the following modifications may be used:

Select a parameter t' which takes on the values from -2 and +2 inclusive for ellipses. (For hyperbolae, the values -2, 0, and +2 are excluded.)

For $t' \geq 0$, we have $t = t' - 1$ and $\sigma = +1$;

For $t' < 0$, we have $t = t' + 1$ and $\sigma = -1$.

This yields for ellipses

$$u = \sigma\frac{2t}{1+t^2} r_u; \qquad v = \sigma\frac{1-t^2}{1+t^2} r_v. \tag{31}$$

and hyperbolae

$$u = \sigma\frac{2t}{1-t^2} r_u; \qquad v = \sigma\frac{1+t^2}{1-t^2} r_v. \tag{32}$$

Since for an ellipse,u and v should be periodic functions, we can set up a parameter t", which is normalized to t' by adding or subtracting multiples of four, such that t' is in the range [-2, +2].

4.5 Non-planar Intersections

The parameterization of planar intersections is described above. For non-planar intersections, the curve lies in a quadric surface which has a "base curve" which is either a line, a parabola, a hyperbola, or an ellipse. This base curve may be a cross-section of the quadric surface, in a plane perpendicular to the main axis. There is one set of straight lines, (one line passing thru each point of the base curve) such that each line lies wholly in the quadric surface, and every point on the quadric surface lies on one of these lines.

One uses parameterization to select a point on the base curve and its corresponding line. By solving a quadratic (second-order) equation, one can then find the intersection of the line with any other quadric surface. In this manner, all the points of the QSIC(s) may be found.

SPECIAL NOTE: THE FOLLOWING ALGORITHM HAS NOT BEEN IMPLEMENTED, AND IS PRESENTED ONLY AS A GUIDE TO A FUTURE IMPLEMENTATION.

V. DATA NEEDED BY THE ALGORITHM

The algorithm needs the following input information in some form:

(1) For each object, it needs to know the dimensions of the "object box", which is a cube or rectangular parallepiped in which the object is contained, as well as the resolution and vector lengths which will be used.

(2) Surface equations must be known for each surface. These may be presented either as ten coefficients (in the form of eq. (1)), or presented as follows:

a) The type of surface (ellipsoid, hyperboloid, cylinder, etc.).

b) The lengths of radii, semi-axes, etc.

c) The orientation and displacement from the origin.

An interactive graphics terminal could be used for specifying a surface and then manipulating and distorting it to suit.

(3) Bounds must be specified for each quadric patch in the same manner as in the Woon algorithm.

(4) Surface intersections must be specified. For each intersection the user must specify to the algorithm the two intersecting patches, the multiplicity of the intersection (how many disjoint parts it has), and whether it is to be a "smooth" or a "sharp" intersection.

A "sharp" intersection abruptly separates two surfaces, providing a clear edge which can be seen if viewed from a proper angle.

"Smooth" intersections are used when several quadric patches are used to approximate a single, higher-order surface. These are not included in drawings unless they occur along limbs. Often the first derivatives will be continuous across a smooth intersection. If one wished to approximate a torus (donut) by using patches of ellipsoids, hyperboloids of one sheet, and cones, smooth intersections would be used.

One usually specifies intersecting surfaces as bounds for each other. However, if there is a smooth intersection with a continuous first derivative, then it may be difficult to tell whether a point is on one side of a boundary or the other. Therefore, the program should automatically compute, for every smooth intersection, another surface from the pencil of the two intersecting surfaces. This surface will act as an auxiliary bounding surface. It should meet both intersecting surfaces at a sharp angle.

VI. CLASSIFYING SURFACE INTERSECTIONS

Consider two intersecting surfaces P and Q. If either is planar, then the intersection is planar. The algorithm described in section VII will apply in these cases.

If neither of the two intersecting surfaces is planar, then one must find the "simplest" surface of the pencil, according to the list of Table III. The simplest surfaces are those in which the subdiscriminant is singular. We first solve the equation:

$$\det(R_u(\alpha)) \;=\; \det(Q_u - \alpha P_u) = 0. \tag{33}$$

In order to insure that this has at least one real root, we will disallow the case where $|P_u|$ is singular and $|Q_u|$ is non-singular. (If this should happen, we interchange them.) Equation (33) is of (at most) third order. This may be written as:

$$-\det(P_u)\alpha^3 + K_2\alpha^2 - K_1\alpha + \det(Q_u) = 0, \tag{34}$$

with K_2 being the sum of the determinants of the combinations of two columns of P_u and one of Q_u, and K_1 being the sum of the determinants of the combinations of one column of P_u and two columns of Q_u.

Once a single root is found, synthetic division may be used to find the other real roots, if they exist.

As pointed out in [Woon 1970, pp. 34-36] $R(\alpha)$ is planar when $\det(R(\alpha)) = 0$ and the sum of the 3 x 3 principal minors of $R(\alpha)$ also vanishes. (This last condition may be expressed as: $D_3(R(\alpha)) = 0$.)

These conditions may only occur when $\det(R_u(\alpha)) = 0$. Therefore, each root in α is tested to see whether $R(\alpha)$ is planar. If any $R(\alpha)$ are planar, the simplest is then chosen as a parameterization surface.

If none of the $R(\alpha)$ so far tested is planar, one checks to see whether any one is a hyperbolic paraboloid. If one is, then it is used for parameterization (see Section 8.4). If not, then one of them may be a cylinder, which would then be used for parameterization.

If, at any point, an *INVALID* $R(\alpha)$ is found, then the two surfaces do not intersect.

Table III contains a list of the surfaces which may be used for parameterization. The simplest surfaces are near the top of the list.

PLANAR	LINE (Imaginary Intersecting Planes)
	Single Plane
	Coincident Planes
	Parallel Planes
	Intersecting Planes
NON-SINGULAR:	Hyperbolic Paraboloids
NON-PLANAR SINGULAR:	Parabolic Cylinder
	Elliptic Cylinder
	Hyperbolic Cylinder
	Cone

Table III. List of Surfaces Which May Be Used for Parameterization

VII. PLANAR INTERSECTIONS

A pencil is planar, if, for some α, $\det(R(\alpha)) = 0$, and $D_3(R(\alpha)) = 0$.

7.1 Finding and Classifying Intersections

First one must determine the nature of the intersection (according to Table I) and transform the planar surface into canonical form. Then one selects some surface of the pencil, other than $R(\alpha)$, and applies the same transofrmation to it. This surface will usually be either P or Q.

a) If the intersection lies in a LINE, then, in the space of the canonical form, this line is the w-axis. In this case, one cancels the first two rows and the first two columns of the discriminant of the other surface, getting a 2 x 2 matrix of the form $\begin{pmatrix} C & J \\ J & K \end{pmatrix}$.

If $C = J = K = 0$, then the line is the intersection.

If $C = J = 0$ and $K \neq 0$, then there is no intersection.

Otherwise, taking $\mathcal{D} = J^2-CK$, we have:

No intersection if $J^2-CK < 0$;

One intersection point if $J^2-CK = 0$; or

Two intersection points if $J^2-CK > 0$.

b) If the intersection lies in a single plane, or in two coincident planes, the plane is the u-v plane in the space of the canonical form. One takes the discriminant of the other surface, cancels out the third row and third column, and compares the resultant 3 x 3 conic section discriminant against Table II to find the nature of the intersection. If INVALID, there is no intersection.

c) If the intersection lies in two planes, one must factor the diplanar intersection into its two constituent planes. If there are two parallel planes, we have an equation $Au^2 + K = 0$, with $AK < 0$, which is factored into $\sqrt{|A|}\,u \pm \sqrt{|K|} = 0$.

If there are two intersecting planes, we have $Au^2 + Bv^2 = 0$, with $AB < 0$, which is factored into $\sqrt{|A|}\,u \pm \sqrt{|B|}\,v = 0$.

Then, each of these planes must be put into single-plane canonical form, making the appropriate transformation to the other surface for each plane. Finally each plane must be processed separately, as above.

If it should happen that a planar intersection consists of two straight lines, these must be processed separately.

7.2 Parameterization

After the nature of the planar intersection is found, one has (unless the intersection was a LINE), a conic section in the u-v-plane. One then rotates the u-v plane so that the axes are the axes of the conic. For a parabola, the v-axis is the axis of symmetry. For a hyperbola, the u-axis separates the two parts. One must, as always, make appropriate transformation of both the other surface and of the transformation matrix, $\mathcal{T}$. One then applies the parameterization, as outlined in section IV.

When actually tracing points, one is in x-y-z space, not u-v-w space. Therefore, it would be more efficient to get the x-y-z coordinates of a point directly, without having to transform coordinates each time, and without taking up primary memory space to store the transformation matrix. The form outlined below accomplishes just this.

$$
\begin{aligned}
x &= (a_x t^2 + b_x t + c_x)/\delta + d_x \\
y &= (a_y t^2 + b_y t + c_y)/\delta + d_y \\
z &= (a_z t^2 + b_z t + c_z)/\delta + d_z
\end{aligned}
\tag{35}
$$

where $\delta = 1$ for a parabola or a line;

$\delta = 1 + t^2$ for an ellipse; and

$\delta = 1 - t^2$ for a hyperbola.

Using this form, one may perform the transformation once, while setting up the coefficients (a, b, c, d).

For completeness, one may use the forms outlined in section 4.4. Table IV contains a FORTRAN routine to do this.

7.3 Finding Parameter Limits

Not all of a Quadric Surface Intersection Curve (QSIC) is part of an edge. Some or all of a QSIC may be out of bounds for one or both of the intersecting surfaces. Since, for planar intersections, each value of the parameter represents a unique point, the endpoints of an edge may be represented as parameter values.

Each QSIC has a multiplicity, which is the number of distinct edges which are intersection curves of the same two surface patches. An example of this would be two cones, back-to-back, with a thin circular cylinder dividing their intersection into two parts. This intersection, between the two cones, has a multiplicity of two.

The intersection of a sphere with a cylinder of diameter less than that of the sphere, such that the center of the sphere is along the axis of the cylinder, is not a single intersection of multiplicity two, but

Note: In the following program, x, y, and z are X(1), X(2), and X(3); a_x is A(1,x), b_x is A(2,x), c_x is A(3,x), and d_x is A(4,x); t is T, and t' and t" are TP.

```
      SUBROUTINE PLANAR (TP, A, ITYPE, X)
C   TP IS THE PARAMETER,  A  IS THE COEFF. MATRIX, &
C   X IS THE RETURNED CO-ORDINATE LOCATION.
C   ITYPE = -1  HYPERBOLA
C            0  PARABOLA OR LINE
C           +1  ELLIPSE
      REAL A(4,3)  X(3)
      IF  (ITYPE) 2, 3, 1
CENTRAL CONICS
   1  TP = TP - 4*INT(TP+SIGN(2,TP))/4)
   2  T = TP - SIGN(1.,TP)
      DENOM = SIGN (1. + ITYPE*T*T, TP)
      GO TO 4
C-- PARABOLA OR LINE
   3  T = TP
      DENOM = 1.
COMPUTE
   4  DO 5  I = 1, 3
   5  X(I) = (A(1,I)*T*T+A(2,I)*T+A(3,I))/DENOM + A(4,I)
      RETURN
      END
```

Table IV. FORTRAN routine for solution of eq. (35)

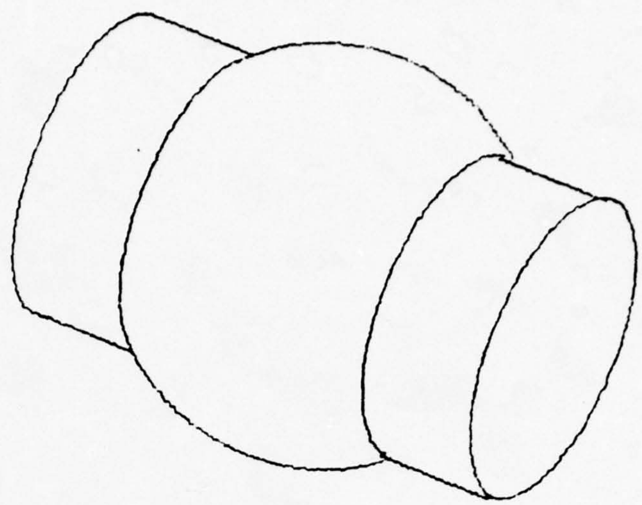

Figure 1: one for each

Intersection.

rather two separate intersections of multiplicity one, as the two parts of the cylinder are two separate patches (see Fig. 1).

A QSIC will have one pair of parameter limits for each unit of multiplicity.

If a quadric surface intersection is a point or pair of points, then any points which are out-of-bounds must be rejected. Otherwise, parameter limits must be found. The program must find one or more ranges of the parameter for which the parameter is continuously within bounds for both of the intersecting surfaces.

First the program establishes a list of bounding surfaces for the QSIC. Then a scan is made of the possible values of the parameter, searching for a value whose point is within bounds.

For an ellipse, the parameter is taken from -2 to +2 (in t' form), with parameter increments of about 0.1. For a parabola or hyperbola, the limits are the values of the parameter at the limits of the object box, but the increments for the hyperbola should be about 0.1 also.

As this is done, the value of $q_b(\vec{x}) = \vec{x}Q_b\,\vec{x}^T$ for each bounding surface Q_b is saved. If no point satisfying the bounds is found in the initial scan, then one may look for "dips"*in the values of the $q_b(\vec{x})$'s. This way one may find a small intersection which was overlooked before. Interval halving is performed at each dip to see whether it results in a previously missed small intersection.

Once a point within bounds is found, one can begin looking for the edge's endpoints. Basically, this is tracing the edge, for different values of the parameter, until a point out of bounds is found. (Often the first point within bounds has an already-traced neighboring point out of bounds.) When such a pair of points is found, one may use interval halving to locate the precise end point. If it is known which bounding surface function changes sign to cause one point to be in bounds and the other out of bounds, one may use Newton's method on that "critical" bounding surface function.

If one is looking for another edge on a particular plane and cannot find one within the parameter values not included in the range of the first edge, it is possible that the first edge is really both edges, with a small gap in between. Then one should go back, checking all points where the value of any bounding surface function changes sign, to see whether these points are endpoints of such a small gap. If none is found, it is likely that the user made an error.

Special note on ellipses: If there are no bounds on an ellipse that cause parameter limits, then the limits are <-2,+2>. If the point $t' = -2(+2)$ is in the middle of an edge, using the t" format allows one to extend paramater limits above +2 or below -2.

---

* An extremum towards zero.

VIII. NON-PLANAR QSICS

A QSIC not contained in a plane is non-planar or twisted. Appendix A shows that such a QSIC is always located in a quadric surface with at least one set of non-intersecting straight lines passing through every point on the surface.

This section shows how to utilize this property to parameterize a non-planar QSIC. Given a parameter, one may find a point (or pair of points) by solving an equation of no more than second-degree. Since the quadratic equations is easy to solve, this method gives the exact location of each point. When there are two points for one parameter value, then one is associated with an increasing parameter, and the other with a decreasing parameter.

8.1 Basic Techniques

A non-planar QSIC is contained in a quadric surface which shall be called the "parameterization surface". It is one of the surfaces of the pencil of the two intersecting surfaces. Table III contains a list of these.

Selecting the $R(\alpha)$ which is "easiest" to handle, we find the proper parameterization for its base curve. For a cylinder, the base curve is a cross-section of the parameterization surface, in a plane perpendicular to the main axis.

Given a parameter one finds the corresponding point on the base curve using eq's. (32). These coordinates are referred to as x_0, y_0, and z_0.

One then selects any member of the pencil other than the parameterization surface. Usually this will be P or Q. One finds the straight line in the parameterization surface which passes through point (x_0, y_0, z_0), and solves for the intersection of the line and the other surface. This will be a second-order equation.

Suppose γ_x, γ_y, and γ_z are the direction cosines of the line corresponding to the base curve point x_0, y_0, z_0. The location of any point on the line is given by

$$\begin{aligned} x &= x_0 + \gamma_x s \\ y &= y_0 + \gamma_y s \\ z &= z_0 + \gamma_z s \end{aligned} \tag{36}$$

where s is a secondary parameter. If the other surface is surface Q and has an equation of the form of eq. (1), then the second-order equation in s is of the form:

$$a\, s^2 + b\, s + c = 0 \tag{37}$$

where

$$a = q_1\gamma_x^2 + q_2\gamma_y^2 + q_3\gamma_z^2 + q_4\gamma_x\gamma_y + q_5\gamma_y\gamma_z + q_6\gamma_z\gamma_x$$

$$\begin{aligned} b = {} & 2q_1x_0\gamma_x + 2q_2y_0\gamma_y + 2q_3z_0\gamma_z + q_4(x_0\gamma_y+y_0\gamma_x) + q_5(y_0\gamma_z+z_0\gamma_y) \\ & + q_6(z_0\gamma_x + x_0\gamma_z) + q_7\gamma_x + q_8\gamma_y + q_9\gamma_z \end{aligned}$$

$$\begin{aligned} c = {} & q_1x_0^2 + q_2y_0^2 + q_3z_0^2 + q_4x_0y_0 + q_5y_0z_0 + q_6z_0x_0 + q_7x_0 \\ & + q_8y_0 + q_9z_0 + q_0 = q(x_0,y_0,z_0) \end{aligned}$$

The quadratic discriminant (not to be confused with the discriminant matrix of a surface) is $\mathcal{D} = b^2 - 4ac$. If $\mathcal{D} < 0$, there are no real roots in s, and the corresponding value of t is actually invalid. If $\mathcal{D} = 0$, then there is one root in s. If $\mathcal{D} > 0$, s has two real roots. There are as many intersection points along the line as there are roots in s. When tracing a QSIC, one uses the larger value of s when the parameter is increasing, and the smaller value of s when the parameter is decreasing. This is discussed in more detail in *section IX.*

8.2 Cylindric Intersections

For cylindric intersections, the line through each point is parallel to the axis of the cylinder, which is the w-axis. One uses the substitution

$$\gamma_x = e_x; \quad \gamma_y = e_y; \quad \gamma_z = e_z;$$

with $(e_x\ e_y\ e_z) = (\ 0\ 0\ 1)\,\mathcal{F}_u$.

The cross section of a cylindric intersection may be taken anywhere along the main axis but the u-v plane is simplest.

8.3 OSICs Lying in Hyperbolic Paraboloids

A hyperbolic paraboloid has two sets of straight lines which lie totally in the surface. These are called reguli. Each regulus is a set of mutually skew lines, and each point on the paraboloid is the intersection of one line from each regulus.

To find the coefficients of parameterization, one takes the hyperbolic paraboloid P and the other surface Q, and transforms them both by a congruence transformation so that P is in canonical form:

$$P = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & J \\ 0 & 0 & J & 0 \end{pmatrix} \tag{38}$$

If A is negative, reverse the sign of every element in P.

Now transform both P and Q by the following valid transformation, which is a product of a rotation and a scaling transformation. It will distort the surfaces but not change their nature:

$$\begin{pmatrix} \sqrt{1/2A} & -\sqrt{1/2B} & 0 & 0 \\ \sqrt{1/2A} & \sqrt{1/2B} & 0 & 0 \\ 0 & 0 & -1/J & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The result is that Q is in the form

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

This is equivalent of the equation (in u-v-w space) uv=w. If t is the primary parameter and s is the secondary parameter, u=t, v=s, and w=st.

In the x-y-z coordinate system, we may express the coordinates in terms of equations (36) as such:

$$\begin{aligned} x &= (c_x t + d_x) + (e_x + f_x t)s \\ y &= (c_y t + d_y) + (e_y + f_y t)s \\ z &= (c_z t + d_z) + (e_z + f_z t)s \end{aligned} \qquad (39)$$

If $\mathcal{T}$ is the transformation matrix from u-v-w space to x-y-z space, then the co-efficients are:

$$\begin{bmatrix} c_x & c_y & c_z & 0 \\ e_x & e_y & e_z & 0 \\ f_x & f_y & f_z & 0 \\ d_x & d_y & d_z & 1 \end{bmatrix} = \mathcal{T} \qquad (40)$$

IX. PARAMETER LIMITS AND QSIC TRACING

9.1 Basic Parameter Limits

For planar QSICs, parameter limits are defined in a two-part process. In the first part, the basic limits are found. For an ellipse, the basic limit string is <-2.,+2.>. For hyperbolae and parabolae, which extend to infinity, the basic parameter limits are those defined by points where the curve meets the limit of the object box.

Once basic limits are found, the bounded parameter limits are found, using the process outlined in section 7.3.

For non-planar intersections, finding the basic parameter limits is more complicated. These limits are the limits of the range of values of the parameter such that both the following conditions are met:

a) If the base curve is a line, a parabola, or a hyperbola, the line corresponding to the parameter must not lie totally outside of the object box; and

b) Equation (37) has at least one real root. This is equivalent to saying that the quadratic discriminant $\mathcal{D} = b^2-4ac$ is non-negative. One would find those ranges of the parameter t' for which $\mathcal{D} \geq 0$. The limits of these ranges will be values of t' for $\mathcal{D} = 0$. Of course, if for all values of the parameter t' satisfying condition (a), $\mathcal{D} < 0$, then there is no intersection.

Once the basic limits are found, basic limit strings may be set up. For non-planar intersections, limit strings have three elements, not two. For basic parameter limits, the first element of the limit string is one of the limits, the second element is the other limit, and the third element is the same as the first.

9.2 Tracing the Edge to Find Bounded Parameter Limits

One next traces the edge within the basic parameter limits. The limit string is $<t_1, t_2, t_3>$, with $t_1 = t_3$. First, one traces from t_1 to t_2, and then one traces in the reverse direction from t_2 to $t_3 = t_1$. The two traces use opposite "radical signs" in the solution of eq. (37).

In any trace from t_i to t_{i+1}, if the parameter is increasing ($t_i < t_{i+1}$), one solves eq. (37) with a positive radical sign

$$s = (-b+\sqrt{b^2-4ac})/2a \tag{41}$$

If the parameter is decreasing ($t_i > t_{i+1}$), one solves it with a negative radical sign

$$s = (-b-\sqrt{b^2-4ac})/2a \tag{42}$$

This implicitly gives the radical sign, without it having to be specified explicitly. The radical sign is the "Boolean parameter" mentioned earlier.

As one is tracing the curve, one looks for parameter bounds just as in section 7.3. If the point corresponding to $t' = t_2$ is in bounds, there is a bounded parameter limit string with t_2 as its second element. All other bounded parameter limit strings would have two limits. Since limit strings for non-planar surfaces should have three elements, the second element may be repeated. Whenever $t_i = t_{i+1}$, there is no trace performed.

X. LIMBS

10.1 Defining Limbs

Since quadric surfaces are generally not flat, it is likely that, from a particular viewpoint, the quadric surface may "fold in back of itself". The locus of points where this happens is the limb. This terminology is due to [Comba 1968]. A virtual edge is a segment of a limb.

More precisely, the limb is the locus of points on a surface where the normal to the surface is perpendicular to the line-of-sight. These points must satisfy both these conditions:

$$q(x,y,z) = 0 \qquad \text{and} \qquad p = \vec{s}.\overrightarrow{grad}\ q = 0 \tag{43}$$

where $\vec{s}$ is the line-of-sight vector from the object point to the view-point; and $\overrightarrow{grad}$ q is the surface normal, which may be expressed as a column vector:

$$\overrightarrow{grad}\ q = \begin{pmatrix} 2q_1x + q_4y + q_6z + q_7 \\ 2q_2y + q_4x + q_5z + q_8 \\ 2q_3z + q_6x + q_5y + q_9 \end{pmatrix} \tag{44}$$

10.2 Orthographic Projections

For simplicity, let us assume that the object has already been translated and rotated so that it is now in the picture space. We shall call the axes x, y, and z. For othographic projections, the viewpoint is considered to be on the x-axis at infinity. The line-of-sight vector is, therefore, along the x-axis and its normalized value is (1 0 0). From eq. (43) we have

$$p = \vec{s}.\overrightarrow{\text{grad}}\ q = 2q_1x + q_4y + q_6z + q_7 = 0 \qquad (45)$$

This is a plane. It is called the "polar plane of the surface". It does not exist if $q_1 = q_4 = q_6 = 0$. (A paraboloid whose axis is the x-axis is an example of a surface without a polar plane.)

10.3 Perspective Projections

Woon did not discuss polar planes for perspective drawings, and it happens that they do require a different equation than orthographic drawings do. Assume that the viewpoint is on the x-axis, a distance D from the origin. We make the simple transformation x' = x - D, putting the viewpoint at the x'-y-z origin. The line-of-sight vector thus becomes $\vec{s} = (-x' \ -y \ -z)$.

Therefore, the equation for the polar surface is

$$p = \vec{s}.\overrightarrow{\text{grad}}\ q$$
$$= -2q'_1x^2-2q_2y^2-2q_3z^2-2q'_4xy-2q_5yz-2q'_6zx'-q'_7x'-q_8y-q_9z=0 \qquad (46)$$

By substituting twice the value of q(x,y,z) from eq. (1), we have

$$p = -2q'(x',y,z) + q'_7x' + q_8y + q_9z + 2q_0 = 0 \qquad (47)$$

Since all points on the limb must be on the surface, with $q'(x',y,z) = 0$, the above equation reduces to:

$$p_v = q'_7x + q_8y + q_9z + 2q_0, \qquad (48)$$

where $p_v = 0$ is the equation of the "virtual polar plane".

10.4 Point Orientation

The function $p(x,y,z)$ has another use besides acting as the equation of the polar plane. It also indicates the orientation of a point on the surface. If $p > 0$, the angle between $\vec{s}$ and the normal is less than $90°$, and so the point is front-oriented and potentially visible. If $p < 0$, then the angle is greater than $90°$ and the point is back-oriented and, therefore, invisible. Of course, if $p = 0$, then the angle is $90°$, and the point is "orthogonally" oriented and is on a limb.

For perspective drawings the function p_v may be used in place of p, since all points that ever need to be tested by the value of p are on the surface anyway.

XI. HIDDEN-LINE DETERMINATION AND PICTURE DRAWING

This algorithm is primarily desgined to handle QSICs, with hidden-line elimination a secondary consideration. The methods described here may be considered "brute-force" methods, but they should work well.

11.1 Preliminary Processing

Each quadric patch is considered, and, if it contains a limb within its bounds, it is divided into two or more faces at the limb. If there is no limb, then the patch is a single face.

Each face is tested to see whether it is a front-face or a back-face. As Woon pointed out, [Woon 1970, p. 26], a face has the same orientation as any point on it.

For an edge to be visible at least one of the faces adjacent to it must be a front-face. This algorithm does not use Loutrel's complete classification system [Loutrel 1970] because it is dependent on whether a surface intersection is obtrusive or intrusive, and a QSIC may be both.

The parameter limit strings of an edge may have to be changed. If an edge is entirely the intersection of two back-faces, it will not be displayed at all in a hidden-line eliminated drawing. If it is the intersection of two quadric patches with limbs, some of it may be the intersection of two back-faces and be hidden, while the rest of it may be potentially visible. The limits must be appropriately reset so that only potentially visible parts are included.

At this point, two comments are in order. First, limbs will now be considered to be like any other intersection, with parameter limits and coefficients. Second, the following discussion is for orthographic drawings only. For perspective drawings, one must substitute "y/x" for "y" and "z/x" for "z", wherever these appear.

For each potentially visible edge, the y- and z-extrema should be computed. (The x-extremum may also be computed.) This may be done by tracing through an edge, looking for places where each coordinate has an extremum. This will happen when the sign of the increment of the coordinate changes. This method is like looking for "dips".

For each front-face, the extrema of the extrema of the associated edges are taken. Thus we have both edge extrema, and face extrema.

11.2 Tracing an Edge

First, for each edge one must find those front faces which may hide all or part of the edge. This is done by using the edge extrema and the face extrema. A face may hide an edge only if the y-extrema and the z-extrema overlap. (If x-extrema are calculated, the nearer x-extremum of the face must be closer to the viewpoint than the further x-extremum of the edge for that edge to be potentially hidden by the face.)

One is now ready to trace the edge, simultaneously detecting which parts are hidden, and drawing the edge. As soon as one is sure that the vector from one point to the next is completely unhidden (hidden), one may draw (skip) that vector. Thus both hidden-line detection and edge drawing occur as parts of the same process.

There are two methods of scanning an edge and they are outlined below. Either may be used.

The "tight" scan uses small vectors. As each point is scanned, one computes the line of sight from that point (test point) to the vantage point. One then computes the intersection of this line with the quadric surface associated with each front face which may hide the point. This is a second-order equation, similar to eq. (37). If there are no roots, the point is not hidden by that face. If there are two roots, one of them corresponds to a back-face, and, since a back-face cannot hide a point, this root is discarded.

One takes the (remaining) root, and checks to see if the corresponding point is within bounds for that surface patch. If it is, the point is hidden if the intersection point is between the test point and the view point. If the intersection point is out of bounds, or is further from the view point than the test point, then the test point is tested against the next face. If no face hides the point, then it is visible.

If one point is hidden and one of its neighbors is not, then there is some point between them where the edge becomes hidden. By using interval-halving, one may locate this point.

This method uses many small vectors to insure that a small hidden (unhidden) section of the edge is not considered visible (hidden). The disadvantage of this method is that many points have to be inspected for each edge.

The "loose" scan uses larger vectors but needs to store a good deal of information about each point. This is all information that would have to be computed anyway.

For each front face which could hide part of the edge, a list called the "state list" is made of the values of the following functions

1) The quadratic discriminant ($\mathcal{D} = b^2 - 4ac$) of the second-degree equation of the face's surface along the line of sight, and

2) The bounding surface function $q_b(x_i, y_i, z_i)$ of each bounding surface of the patch on which the face is located, where x_i, y_i, and z_i are the coordinates of the intersection point.

If there is a change of sign of any function in any state list, then it is possible that, at some intermediate point, the edge becomes hidden. One may use interval halving the locate this point. It is also helpful to look for "dips" in these functions.

The "loose" scan gives an effective resolution which is much better than the vector length actually used.

11.3 Vector Length

In using the "loose" scan, one may use a variable vector length. If $\vec{p}$ and $\vec{q}$ are two adjacent vectors, then $|\vec{p}| x |\vec{q}| = |\vec{p}| |\vec{q}| \sin pq$. By trying to equalize $|\vec{p}| x |\vec{q}|$ one should get an optimal balance between having fewest vectors (requiring least processing and/or storage space) and making sure no angles are too sharp. Using this equalization, the parts of a curve with the greatest curvature will have the shortest vectors and the largest inter-vector angles, while nearly straight curves would have long vectors and small angles. However, because of the requirements of the algorithm in looking for sign changes and "dips", each edge should have at least two or three vectors. It might also be good to have an upper limit on vector length.

One equalizes the cross product by using a variable parameter increment. Suppose we have a lower limit (L_i) and an upper limit (L_u) on the cross product $\pi_c = |\bar{p}|x|\bar{q}|$. These may be in a ratio of 1:1.5 or 1:2. If π_c falls below L_i, then one increases the parameter increment. If π_c rises above L_u, then the parameter increment is reduced.

XII. CONCLUSION

The chief advantages of this algorithm are (1) it is a complete quadric surface algorithm, allowing use of all real quadric surfaces; (2) it allows "smooth" intersections to approximate higher-order surfaces, and (3) it may be fast enough to be used in conjunction with a shading algorithm such as outlined in [Phong 1975].

Since this algorithm distinguishes limbs, sharp edges, and smooth edges, each can be appropriately handled by the shading algorithm. This can include explicit handling of the mach band effect, specular reflection, and transparency [Metelli 1974].

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APPENDIX 1 - PROOF OF THEOREM

In this report it has been assumed that all QSICs lie either in singular quadric surfaces or else in hyperbolic paraboloids. The following is a proof of this conjecture.

DEFINITIONS

A quadric surface is positive, singular, or negative if the determinant of its discriminant matrix is positive, zero, or negative respectively. A pencil of quadric surfaces is singular iff it contains at least one singular quadric surface, and is non-singular otherwise. A surface intersection (QSIC) is singular if the pencil of the two intersection surfaces is singular, and is non-singular if the pencil is non-singular.

A congruence transformation is a non-singular transformation such that the upper-left 3 x 3 submatrix of the transformation matrix (called the rotational part) is orthogonal, and the last column of the matrix is $(0\ 0\ 0\ 1)^T$.

A scaling transformation is a non-singular transformation of the form:

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \qquad \text{with } \alpha, \beta, \gamma, \delta \neq 0.$$

A valid transformation is the product of congruence and scaling transformations.

Notice that valid transformations do not change the type of surface the discriminant describes, although scaling transformations may distort such a surface.

DEFINITION:

A PARA is a quadric surface with a singular sub-discriminant. PARAs include all singular quadric surfaces except cones, as well as elliptic and hyperbolic paraboloids.

LEMMA 1:

The pencil of any two quadric surfaces contains a PARA.

Proof: Suppose we have two quadric surfaces, P and Q.

[Case 1: Either P or Q is a PARA. The hypothesis is satisfied.

[Case 2: Neither P nor Q is a PARA. The equation:

$\det(R_u(\alpha)) = \det(Q_u - \alpha P_u) = 0$ is of the form:

$$-\det(P_u)\alpha^3 + K_2\alpha^2 - K_1\alpha + \det(Q_u) = 0,$$

with K_2 and K_1 being the sums of the determinants of combinations of the columns of P_u and Q_u. Because $\det(P_u) \neq 0$ and $\det(Q_u) \neq 0$, this equation is of third order and no less. Since a third order equation must have at least one real root, there is at least one value of α such that $\det(R(\alpha)) = 0$. This $R(\alpha)$ is, by definition, a PARA.

LEMMA 2:

Any pencil of a positive quadric surface and a negative quadric surface is singular.

Proof: The equation $\det(R(\alpha)) = \det(Q-\alpha P) = 0$ is equivalent to $\det(P)\alpha^4 - K_3\alpha^3 + K_2\alpha^2 - K_1\alpha + \det(Q) = 0$, where K_3, K_2, and K_1 are sums of determinants of combinations of columns of P and Q. If P is positive and Q is negative, then $\det(R(0)) = \det(Q) < 0$; while for $\alpha \to \pm\infty$, $\det(R(\alpha)) \to +\infty > 0$. Because $R(\alpha)$ is a continuous function, there must be at least two values of α, one positive and one negative, for which $R(\alpha)$ is singular.

LEMMA 3:

One may apply any valid transformation to any pair of quadric surfaces without affecting the roots of the equations: $\det(Q-\alpha P) = 0$ and $\det(Q_u-\beta P_u) = 0$.

Proof: Suppose we have the valid transformation S and its rotational part, S_u. Both S and S_u are non-singular. If $\Sigma = S^{-1}$, then $\Sigma_u = S_u^{-1}$, and both S_u and Σ_u are non-singular.

$$\mathrm{Det}(SQ\Sigma-\alpha SP\Sigma) = \det(S(Q-\alpha P)\Sigma) = \det(S)\det(Q-\alpha P)\det(\Sigma)$$

$= \det(S)\det(\Sigma)\det(Q-\alpha P) = \det(Q-\alpha P)$. Similarly for Q_u, P_u, S_u, Σ_u and β.

LEMMA 4:

The arbitrary elliptic paraboloid P and the arbitrary quadric surface Q may be expressed, in some transformed space, as:

$$Q = \begin{pmatrix} A & D & 0 & G \\ D & B & E & H \\ 0 & E & C & J \\ G & H & J & K \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 0 & 1 \\ X & Y & 1 & 0 \end{pmatrix}$$

Proof: First, let us select the axes such that the z-axis is the main axis of the paraboloid P, the origin is located on P, and the x- and y-axes are the axes of the ellipse which the cross-section of P. This puts the surfaces' discriminants in the form:

$$Q'' = \begin{pmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{pmatrix} \qquad P'' = \begin{pmatrix} S & 0 & 0 & X \\ 0 & T & 0 & Y \\ 0 & 0 & 0 & Z \\ X & Y & Z & 0 \end{pmatrix}$$

If S is negative, reverse the sign of every element of P".

Now, use the following scaling transformation on both P" and Q":

$$\begin{pmatrix} \sqrt{1/S} & 0 & 0 & 0 \\ 0 & \sqrt{1/T} & 0 & 0 \\ 0 & 0 & 1/Z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This gives us:

$$Q' = \begin{pmatrix} A & D & F & G \\ D & B & E & H \\ F & E & C & J \\ G & H & J & K \end{pmatrix} \qquad P' = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 0 & 1 \\ X & Y & 1 & 0 \end{pmatrix}$$

We now apply the following rotation (which is a congruence transformation) to both P' and Q':

$$\begin{pmatrix} L & -M & 0 & 0 \\ M & L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with } L^2 + M^2 = 1 \text{ and } \frac{L}{M} = \frac{E}{F}.$$

Using this transformation yields:

$$Q = \begin{pmatrix} A & D & 0 & G \\ D & B & E & H \\ 0 & E & C & J \\ G & H & J & K \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 0 & 1 \\ X & Y & 1 & 0 \end{pmatrix}$$

LEMMA 5:

The pencil of two elliptic paraboloids is always singular.

Proof: Suppose we have two elliptic paraboloids P and Q, in the form given in LEMMA 4. There are some conditions that are imposed on the elements of Q, as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 0 & 1 \\ X & Y & 1 & 0 \end{pmatrix} \qquad Q = \begin{pmatrix} A & D & 0 & G \\ D & B & E & H \\ 0 & E & C & J \\ G & H & J & K \end{pmatrix}$$

$\det(P_u) = 0$ such that: $\det(Q_u) = 0$,

$T_2(P) > 0$ $T_2(Q) > 0$.

The two conditions for Q are:

$\det(Q_u) = ABC - CD^2 - AE^2 = 0$;

$T_2(Q) = AB + BC + AC - D^2 - E^2 > 0$.

[Case 1: $C=0$: $\det(Q_u) = 0$ only if $A = 0$ or $E = 0$.

[Case 1a: $C = E = 0$

$$Q = \begin{pmatrix} A & D & 0 & G \\ D & B & 0 & H \\ 0 & 0 & 0 & J \\ G & H & J & K \end{pmatrix}.$$

Obviously, $\det(Q-JP) = 0$.

[Case 1b: $A = C = 0$.

$$Q = \begin{pmatrix} 0 & D & 0 & G \\ D & B & E & H \\ 0 & E & 0 & J \\ G & H & J & K \end{pmatrix}$$

Now, $T_2(Q) = -D^2 - E^2 \leq 0$. However, one of the conditions for Q to be an elliptic paraboloid is that $T_2 > 0$. Thus, case 1b never occurs.

[Case 2: $C \neq 0$. For simplicity, we may divide every element in Q by C. This does not change the surface.

$$Q = \begin{pmatrix} A & D & 0 & G \\ D & B & E & H \\ 0 & E & 1 & J \\ G & H & J & K \end{pmatrix}.$$

Now, $\det(Q_u) = AB - AE^2 - D^2 = 0$, so $D = \pm\sqrt{A(B-E^2)}$.

For D to be real, we must have:

$$A(B-E^2) \geq 0 \qquad \text{(condition a)}$$

We also have: $T_2(Q) = AB + A + B - D^2 - E^2 > 0$.

Since: $D^2 = AB - AE^2$, we have:

$$T_2(Q) = A + B - E^2 + AE^2 \qquad \text{(condition b)}$$

Now, taking the pencil of P and Q: $R(\alpha) = (Q-\alpha P)$; solving $\det(R_u(\alpha)) = \det(Q_u-\alpha P_u) = 0$, we get:

$$-\det(P_u)\alpha^3 + \alpha^2 - (A+B-E^2)\alpha + \det(Q_u) = 0.$$

Since $\det(P_u) = \det(Q_u) = 0$, when we reject the root $\alpha = 0$, we have the remaining root: $\alpha = A + B - E^2$. For this α, $R(\alpha)$ is a PARA, by definition.

[Case 2a: If $R(\alpha)$ is a singular PARA, then the hypothesis is satisfied.

[Case 2b: If $R(\alpha)$ is not a singular PARA, it must be a paraboloid, either elliptic or hyperbolic. We will assume that it is elliptic. Then $T_2(Q) > 0$.

Substituting the root of α as above, we have:

$$R_u(\alpha) = (Q_u-\alpha P_u) = \begin{pmatrix} E^2-B & D & 0 \\ D & E^2-A & E \\ 0 & E & 1 \end{pmatrix}.$$

Taking the condition that $T_2(R(\alpha)) > 0$: (Taking $D^2 = AB - AE^2$)

$$T_2 = (E^2-A)(E^2-B) + 2E^2 - (A+B) - AB + AE^2 > 0;$$

$$T_2 = E^2(E^2-B) + E^2 - A - B > 0; \text{ or}$$

$$-T_2 = E^2(B-E^2) + A + B - E^2 < 0.$$

Now, taking $\beta = B - E^2$, the three conditions are:

$$A\beta \geq 0 \qquad \text{(condition a)}$$

$$A + \beta + AE^2 = A(1+E^2) + \beta > 0 \qquad \text{(condition b)}$$

$$E^2\beta + A + \beta = A + \beta(1+E^2) < 0 \qquad \text{(condition c)}$$

By (a), if $\beta < 0$, then $A < 0$; if $A < 0$, then $\beta < 0$;

if $\beta > 0$, then $A > 0$; if $A > 0$, then $\beta > 0$.

By (b), if $\beta \leq 0$, then $A > 0$; if $A \leq 0$, then $\beta > 0$.

By (c), if $\beta \geq 0$, then $A < 0$; if $A \geq 0$, then $\beta < 0$.

These conditions are inconsistent; hence $R(\alpha)$ cannot be an elliptic paraboloid and must be hyperbolic paraboloid. Because this is a pencil of two negative elliptic paraboloids and a positive hyperbolic paraboloid, by LEMMA 2, the pencil and its associated QSIC are singular.

LEMMA 6:

The pencil of two negative quadric surfaces is singular.

Proof: Suppose we have two negative quadric surfaces P and Q. By Lemma 1, the pencil contains a PARA. If the PARA is singular, then so is the pencil and the QSIC. If not, then the PARA is a paraboloid. If the PARA is a (positive) hyperbolic paraboloid, then the pencil is singular by LEMMA 2. If there is more than one PARA, then these tests may be applied to each of them.

Otherwise, the PARA is an elliptic paraboloid. Let us call the PARA P, and choose one of the two negative quadric surfaces Q. If Q is another elliptic paraboloid, then the pencil is singular by LEMMA 5. Otherwise, Q must be an ellipsoid or a hyperboloid of two sheets, with $\det(Q_u) \neq 0$.

By LEMMA 4, the discriminants of P and Q may be expressed:

$$Q = \begin{pmatrix} A & D & O & G \\ D & B & E & H \\ O & E & C & J \\ G & H & J & K \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 0 & 1 \\ X & Y & 1 & 0 \end{pmatrix}$$

The solution of the equation: $\det(R_u(\alpha)) = \det(Q_u - \alpha P_u) = 0$ is:

$$-\det(P_u)\alpha^2 + C\alpha^2 - (BC-E^2+AC)\alpha + \det(Q_u) = 0; \text{ or}$$

$$C\alpha^2 - [(A+B)C-E^2]\alpha + [(AB-D^2)C-AE^2] = 0 \tag{A1}$$

[Case 1: $C = 0$. Equation (A1) gives $\alpha = A$.

$$R_u(A) = (Q_u - AP_u) = \begin{pmatrix} O & D & O \\ D & B-A & E \\ O & E & O \end{pmatrix}$$

$$T_2(R(A)) = -D^2-E^2 \leq 0.$$

If $D = E = 0$, then Rank $(R_u(A))$ is at most one. Since the rank of the discriminant cannot exceed the rank of the sub-dsicriminant by more than two, Rank $(R(A)) \leq 3$, so $R(A)$ is singular, and so is the pencil and the QSIC.

If $D \neq 0$ or $E \neq 0$, then $T_2(R(A)) < 0$, so $R(A))$ is a hyperbolic paraboloid, and, by LEMMA 2, the QSIC is singular.

[Case 2: $C \neq 0$. If we normalize so that $C = 1$, equation (A1) becomes: $\alpha^2 - [(A+B)-E^2]\alpha + [AB-D^2-AE^2] = 0$. Taking the quadratic form:

$a\alpha^2 + b\alpha + c = 0$, we get the quadratic discriminant: $\mathcal{D} = b^2-4ac$.

For this equation, $\mathcal{D} = (A-B+E^2)^2 + 4D^2 \geq 0$.

[Case 2a: If $\mathcal{D} = 0$, then $D = 0$ and $E^2 = B - A$. There is one double root, $\alpha = A$, and Rank $(R_u(A)) \leq 1$, so Rank $(R(A)) \leq 3$, so $R(A)$ is singular, and so is the pencil.

[Case 2b: If $\mathcal{D} > 0$, then there are two roots in α. Both $R(\alpha)$ are PARAs. If one is singular, the hypothesis is satisfied. If one is a positive hyperbolic paraboloid, then the pencil is singular by LEMMA 2. If one is an elliptic paraboloid, the pencil is singular by LEMMA 5.

THEOREM:

The intersection of two quadric surfaces either is singular, lies in a hyperbolic paraboloid, or both.

Proof: If one of the surfaces is singular, the hypothesis is satisfied. If one is positive and one negative, the pencil, and, therefore, the intersection, is singular by LEMMA 2. If both are negative, then the pencil and QSIC are singular by LEMMA 6. If both are positive, then by LEMMA 1, the intersection is contained in a PARA. If the PARA is negative or singular, the intersection is singular by LEMMA 2, or directly. Otherwise, the PARA must be a hyperbolic paraboloid.

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quadric-surface intersection curves

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An algorithm is described for generating two-dimensional, visible-line projections of three-dimensional objects that are bounded by patches of quadric surfaces.

The main task of the algorithm is the calculation of intersections between quadric surfaces. A parameterization scheme is used. Each quadric-surface intersection curve (QSIC) is represented as a set of coefficients

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20. Abstract (Continued)

and parameter limits. Each value of the parameter represents at most two points, and these may easily be distinguished. This scheme can find the coordinates of points of even quartic (fourth-order) intersection curves, using equations of no more than second order.

Methods of parameterization for each type of QSIC are discussed, as well as the problems of surface bounding and hidden-surface removal.